On the multi-component nonlinear Schrödinger equation with constant boundary conditions

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Abstract

The multi-component nonlinear Schrödinger equation related to $\mathbf{C.I} \simeq Sp(2p)/U(p)$ and $\mathbf{D.III} \simeq SO(2p)/U(p)$-type symmetric spaces with non-vanishing boundary conditions is solvable with the inverse scattering method (ISM). As Lax operator L we use the generalized Zakharov-Shabat operator. We show that the ISM for the Lax operator $L(x, \lambda)$ is a nonlinear analog of the Fourier-transform method. As appropriate generalizations of the usual Fourier-exponential functions we use the so-called "squared solutions", which are constructed in terms of the fundamental analytic solutions (FAS) $\chi^{\pm}(x, \lambda)$ of $L(x, \lambda)$ and the Cartan-Weyl basis of the Lie algebra, relevant to the symmetric space. We derive the completeness relation for the "squared solutions" which turns out to provide spectral decomposition of the recursion (generating) operators $\Lambda_{\pm}$, a natural generalizations of $\frac{1}{i}\frac{d}{dx}$ in the case of nonlinear evolution equations (NLEE).

1 Introduction

The integrability of the scalar nonlinear Schrödinger equation (NLS) with vanishing boundary conditions (v.b.c.):

$$iq_t + q_{xx} + 2|q(x,t)|^2 q(x,t) = 0 \tag{1.1}$$

was discovered by Zakharov and Shabat in their pioneer work [21]. Soon after [22] Zakharov and Shabat proved the integrability and the physical importance of the NLS with constant boundary conditions (c.b.c.):

$$iq_t + 2q_{xx} - 2(|q(x,t)|^2 - \rho^2)q(x,t) = 0, \qquad \lim_{x\to\pm\infty} q(x,t) = q_{\pm}, \tag{1.2}$$

where the asymptotic values $q_{\pm}$ satisfy $|q_{\pm}|^2 = \rho^2$. Notice the sign difference in the cubic nonlinearity as well as the additional term with the chemical potential ρ.

Both versions of NLS equation served as models on which generalizations were made. The simplest non-trivial multicomponent generalization of NLS is the vector NLS known as the Manakov model [16]:

$$i\vec{q}_t + \vec{q}_{xx} + 2(\vec{q}^{\,\dagger}\vec{q}(x,t))\vec{q}(x,t) = 0, \tag{1.3}$$

where $\vec{q}(x,t)$ is an n-component complex-valued vector vanishing fast enough for $x \to \pm\infty$. The c.b.c. version of vector NLS

$$i\vec{q}_t + \vec{q}_{xx} - 2\left((\vec{q}^{\,\dagger}\vec{q}(x,t)) - \rho^2\right)\vec{q}(x,t) = 0, \tag{1.4}$$

where $\lim_{x\to\pm\infty} \vec{q}(x,t) = \vec{q}_\pm$ and $\vec{q}_- = U_0 \vec{q}_+$ where U_0 is constant unitary matrix also finds applications. Here $\rho^2 = \vec{q}^\dagger_\pm \vec{q}_\pm$.

Equations (1.1) and (1.3) are particular cases of matrix NLS which is obtained from the system:

$$i\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{qrq}(x,t) = 0 \ , \ -i\mathbf{r}_t + \mathbf{r}_{xx} + 2\mathbf{rqr}(x,t) = 0, \tag{1.5}$$

after imposing appropriate involution (reduction) compatible with the evolution of (1.5). Here $\mathbf{q}$ and $\mathbf{r}$ are $n \times m$ matrix-valued functions of x and t. One such involution is:

$$\mathbf{r} = B_- \mathbf{q}^\dagger B_+^{-1}, \qquad B_\pm = \mathrm{diag}\,(\epsilon_1^\pm, ..., \epsilon_m^\pm), \quad (\epsilon_1^\pm)^2 = 1, \tag{1.6}$$

and the corresponding MNLS acquires the form:

$$i\mathbf{q}_t + \mathbf{q}_{xx} + 2\mathbf{q} B_- \mathbf{q}^\dagger B_+^{-1} \mathbf{q} = 0, \tag{1.7}$$

For $n = m = 1$ and $r = q^*$ the system goes into the scalar NLS (1.1); for $m = 1$ and $n > 1$ and with appropriate choice of involution (1.6) the system is transformed into the Manakov model (1.3). All these versions are solvable with the ISM. The ISM is applicable to nonlinear evolution equations (NLEE) if they can be represented as compatibility condition of two linear problems [18, 20, 1, 2]:

$$[L(\lambda), M(\lambda)] = 0, \tag{1.8}$$

which holds identically with respect to the spectral parameter λ.

The two linear operators $L(\lambda)$ and $M(\lambda)$ in the Zakharov-Shabat system (Z-Sh) for the MNLS on symmetric spaces associated with the simple Lie algebra $\mathfrak{g} \simeq \mathbf{C_r}$ and $\mathfrak{g} \simeq \mathbf{D_r}$ with (v.b.c.) are:

$$L\psi = \left(i\frac{\partial}{\partial x} + Q(x,t) - \lambda\,\sigma_3 \right) \psi(x,t,\lambda) = 0, \tag{1.9}$$

$$M\psi = \left(i\frac{\partial}{\partial t} + V_2(x,t) + \lambda V_1(x,t) - 2\lambda^2 \sigma_3 \right) \psi(x,t,\lambda) = 0, \tag{1.10}$$

$$Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}(x,t) \\ \mathbf{r}(x,t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \tag{1.11}$$

where $Q(x,t)$ and σ_3 are $2r \times 2r$ matrices with compatible block structure. Here

$$V_1(x,t) = 2Q(x,t), \quad V_2(x,t) = [\mathrm{ad}\,_{\sigma_3}^{-1} Q, Q\,] + 2i\mathrm{ad}\,_{\sigma_3}^{-1} Q_x(x,t) \tag{1.12}$$

and $\mathrm{ad}\,_{\sigma_3}^{-1}$ is the inverse of the adjoint action $\mathrm{ad}\,_{\sigma_3}$ with respect to the element σ_3: $\mathrm{ad}\,_{\sigma_3} Y = [\sigma_3, Y]$.

An effective tool to obtain new versions of MNLS is the reduction group introduced by Mikhailov [17]. It allows one to impose algebraic constraints on the potential $Q(x,t)$ which are automatically compatible with the evolution. For example, the involution (1.6), which leads to MNLS with v.b.c. (1.7) is known as $\mathbb{Z}_2$-reduction and can be written as [10]:

$$BU^\dagger(x,t,\lambda^*)B^{-1} = U(x,t,\lambda) \tag{1.13}$$

where B is an automorphism of $\mathfrak{g}$ matrix such that $B^2 = \mathbb{1}$, $[\sigma_3, B] = 0$, and

$$U(x,t,\lambda) = Q(x,t) - \lambda\,\sigma_3. \tag{1.14}$$

Below we analyze the multi-component nonlinear Schödinger equation (MNLS):

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2\mathbf{q}\mathbf{q}^\dagger\,\mathbf{q} + \mathbf{q}\overline{\mu} + \mu\,\mathbf{q} = 0, \tag{1.15}$$

with *constant* boundary conditions (c.b.c.) at $x \to \pm\infty$:

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm\,, \qquad \mu = \mathbf{q}_+\mathbf{q}_+^\dagger = \mathbf{q}_-\mathbf{q}_-^\dagger\,, \qquad \overline{\mu} = \mathbf{q}_+^\dagger\mathbf{q}_+ = \mathbf{q}_-^\dagger\mathbf{q}_-\,, \tag{1.16}$$

where $\mathbf{q}(x,t)$ is $n\times r$ matrix-valued function, related to **A.III**, **C.I** or **D.III**-type symmetric spaces. The case $n \neq r$ can be related to **A.III** symmetric spaces only and has been solved with the ISM in [11].

Therefore we concentrate on the MNLS (1.15) related to **C.I** or **D.III**-type symmetric spaces, which means in particular that $n = r$. Its Lax pair is obtained from (1.9)–(1.12) by replacing $V_2(x,t)$ with:

$$V_2(x,t) = [\mathrm{ad}\,_{\sigma_3}^{-1}Q, Q] + 2i\mathrm{ad}\,_{\sigma_3}^{-1}Q_x(x,t) - \sigma_3 Q_\pm^2. \tag{1.17}$$

Here we have also imposed the additional condition $Q_+^2 = Q_-^2$. It ensures that the two asymptotic Lax operators $L_\pm = i\frac{d}{dx} + Q_\pm - \lambda\sigma_3$ have the same spectrum. It also ensures that the potentials $V_1(x,t)$ and $V_2(x,t)$ in the second operator $M(\lambda)$ vanish for $x \to \pm\infty$. As a result the solutions of the MNLS (1.15) $\mathbf{q}(x,t)$ do not undergo strong oscillations with respect to time, see [11, 9].

Lax operators of the form (1.9) can be associated with each of the symmetric spaces listed below (for the definition see [12] and the Appendix). They are defined by specifying the simple Lie algebra $\mathfrak{g}$, having typical representation in $2r\times 2r$ matrices and the Cartan subalgebra element σ_3:

- **C.I**: $\mathfrak{g} \simeq \mathbf{C}_r \simeq sp(2r)$, $\sigma_3 = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to σ_3 is given by $\vec{a} = \sum_{k=1}^r e_k$
- **D.III** : $\mathfrak{g} \simeq \mathbf{D}_r \simeq so(2r)$, $\sigma_3 = H_{\vec{a}}$, where the vector $\vec{a}$ in the root space $\mathbb{E}^r$ dual to σ_3 is given by $\vec{a} = \sum_{k=1}^r e_k$

Here the orthonormal vectors e_k span the root space $\mathbb{E}^r$ of both types of algebras. The element σ_3 belongs to the Cartan subalgebra $\mathfrak{h}$ and is dual to $\vec{a}$. Using σ_3 we can split the set of positive roots into two two subsets $\triangle^+ = \triangle_0^+ \cup \triangle_1^+$. These sets, for the algebras that we are working with, are composed of the following roots:

$$\triangle_0^+ \equiv \{e_i - e_j\,,\ 1 \le i < j \le r\}\,, \qquad \triangle_1^+ \equiv \{\,2e_i,\, e_i + e_j\,,\ 1 \le i < j \le r\} \tag{1.18}$$

for $\mathfrak{g} \simeq sp(2r)$ and

$$\triangle_0^+ \equiv \{e_i - e_j\,,\ 1 \le i < j \le r\}\,, \qquad \triangle_1^+ \equiv \{e_i + e_j\,,\ 1 \le i < j \le r\} \tag{1.19}$$

for $\mathfrak{g} \simeq so(2r)$.

The root vectors of the algebra are denoted by E_α where α is the corresponding root.

Let us introduce a projector $P_{\sigma_3} = \mathrm{ad}\,_{\sigma_3}^{-1}\mathrm{ad}\,_{\sigma_3}$ onto the co-adjoint orbit O_{σ_3} of the element σ_3. Here the inverse of the adjoint action is $\mathrm{ad}\,_{\sigma_3}^{-1}Y = \frac{1}{2}\sigma_3 Y$. The generic element of $X \in O_{\sigma_3}$ is the one that satisfies the relation $X = P_{\sigma_3}X$. Obviously the potential of the Z-Sh system $Q(x,t)$ and its variation $\delta Q(x,t)$ belong to O_{σ_3}.

This paper extends the results of [11, 9]. In Section 2 we focus on the solutions of the direct scattering problem for the case of Lax operator describing MNLS with c.b.c.. In Section 3 we derive the completeness relation for the "squared solutions" of the Lax operator generalizing the results of [14, 15]. Here we prove that the ISM is equivalent to a generalized Fourier transform also for Lax operators with c.b.c. Thus we have shown that the nonlinear evolution of equation (1.15) transforms into linear one in terms of the scattering data of L.

2 Solutions of the Lax operator L

The spectrum of the asymptotic operators $L_{\pm}$ is purely continuous and is determined by the eigenvalues of $Q_{\pm}$ which generically may be arbitrary complex numbers. However, here we consider only the case when L becomes self-adjoint. As a result its potential $Q(x,t)$ acquires the form:

$$Q(x,t) = -Q^{\dagger}(x,t) \qquad Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}(x,t) \\ -\mathbf{q}^{\dagger}(x,t) & 0 \end{pmatrix} \tag{2.1}$$

For simplicity reasons we will consider only the case when all of the eigenvalues of the asymptotic matrices $Q_{\pm}$ are real and equal:

$$m_1 = m_2 = ... = m_r = m \neq 0 \qquad m \in \mathbb{R} \tag{2.2}$$

As a result we have the following condition on the eigenvalues of the asymptotic matrices [11]: $\mathbf{q}_{\pm}\mathbf{q}_{\pm}^{\dagger}(x,t) = m^2 \mathbb{1}$ and the correspondence with the isotropic problem is obvious: $\mu = \overline{\mu} = m^2\mathbb{1}$.

The requirement that the potentials of the Z-Sh system belong to $\mathfrak{g}$ can be formulated as a reduction condition [17, 5]:

$$S_0^{-1}U^t(x,t,\lambda)S_0 = -U(x,t,\lambda), \quad S_0^{-1}V^t(x,t,\lambda)S_0 = -V(x,t,\lambda), \quad S_0^{-1}\sigma_3 S_0 = -\sigma_3, \tag{2.3}$$

which has trivial action on λ. The matrix S_0 is the one which realizes the definition of the algebras $\mathbf{C}_r \simeq sp(2r)$ or $\mathbf{D}_r \simeq so(2r)$ in the typical representation [5, 12] . In what follows we will define the Lie algebra $\mathfrak{g}$ by:

$$\mathfrak{g} \equiv \left\{ X : X + S_0^{-1}X^t S_0 = 0 \right\}, \tag{2.4}$$

where

$$S_0 = \sum_{s=1}^{r} (-1)^{s+1}(E_{s\bar{s}} - E_{\bar{s}s})$$

for $\mathfrak{g} \simeq sp(2r)$ and

$$S_0 = \sum_{s=1}^{r} (-1)^{s+1}(E_{s\bar{s}} + E_{\bar{s}s})$$

for $\mathfrak{g} \simeq so(2r)$. Here $\bar{s} = 2r - s + 1$ and E_{ks} are $2r \times 2r$ matrices, defined by $(E_{ks})_{ij} = \delta_{ki}\delta_{sj}$. Note that $S_0^2 = \epsilon_0 \mathbb{1}$, where $\epsilon_0 = -1$ for $sp(2r)$ and $\epsilon_0 = 1$ for $so(2r)$.

Such reduction (2.3) imposes restrictions only on the coefficients of $Q(x,t)$ such that for $\mathbf{C}_r \simeq sp(2r)$ we can put:

$$Q(x,t) = \sum_{i<j} \left(q_{ij} E_{e_i+e_j} - q_{j\,i}^{*} E_{-e_i-e_j} \right) + \sum_{i=1}^{r} \left(q_i E_{2e_i} - q_i^{*} E_{-2e_i} \right), \tag{2.5}$$

while in the $\mathbf{D}_r \simeq so(2r)$-case we have:

$$Q(x,t) = \sum_{i<j} \left(q_{ij} E_{e_i+e_j} - q_{j\,i}^{*} E_{-e_i-e_j} \right), \tag{2.6}$$

where * means complex conjugation. The definitions of the root vectors E_α can be found in the Appendix. In the typical representations of $\mathbf{C}_r$ and $\mathbf{D}_r$ these choices for $Q(x,t)$ have always the block structure shown in (2.1). In the case of $\mathfrak{g} \simeq sp(4)$ the block $\mathbf{q}$ is parametrized by three functions:

$$\mathbf{q}(x,t) = \begin{pmatrix} q_{12} & \sqrt{2}q_1 \\ \sqrt{2}q_2 & -q_{12} \end{pmatrix}. \tag{2.7}$$

The corresponding sets of MNLS for these choices of $Q(x,t)$ and v.b.c. were first derived in [5]. For c.b.c. with $\mathbf{r} = -\mathbf{q}^\dagger$ MNLS take the form (1.15) with the additional linear in $\mathbf{q}$ terms ensuring regular behavior of the solutions for $t \to \pm\infty$.

Let us outline the construction of the fundamental analytic solutions (FAS). In the particular case that we are considering - the isotropic problem - the Jost solutions are defined as fundamental solutions with fixed asymptotic for $x \to \pm\infty$:

$$\lim_{x\to\infty} \psi(x,\lambda) e^{ij(\lambda)\sigma_3 x} = \psi_0(\lambda) \quad \lim_{x\to-\infty} \phi(x,\lambda) e^{ij(\lambda)\sigma_3 x} = \phi_0(\lambda), \tag{2.8}$$

where $2r \times 2r$ matrices $\psi_0(\lambda)$ and $\phi_0(\lambda)$ take value in the corresponding group $\mathcal{G}$ and diagonalize the potential of the Lax operator L

$$(Q_+ - \lambda\sigma_3)\,\psi_0(\lambda) = -\psi_0(\lambda) j(\lambda)\sigma_3, \qquad (Q_- - \lambda\sigma_3)\,\phi_0(\lambda) = -\phi_0(\lambda) j(\lambda)\sigma_3, \tag{2.9}$$

where $j(\lambda) = \sqrt{\lambda^2 - m^2}$. They have the block structure:

$$\psi_0(\lambda) = \begin{pmatrix} \underline{A} & S_1 \underline{B} \\ \underline{B}\, S_1 & \underline{A} \end{pmatrix}, \qquad \phi_0(\lambda) = V_0 \begin{pmatrix} \underline{A} & S_1 \underline{B} \\ \underline{B}\, S_1 & \underline{A} \end{pmatrix}. \tag{2.10}$$

The $r \times r$ matrices $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}$ and S_1 are given by:

$$\underline{A}_{\,kl} = \delta_{kl}\sqrt{\frac{\lambda + j(\lambda)}{2j(\lambda)}}, \qquad \underline{B}_{\,kl} = \delta_{kl}\sqrt{\frac{\lambda - j(\lambda)}{2j(\lambda)}}, \qquad S_1 = \sum_{s=1}^{r} (-1)^{s+1} e_{\,s\,,r-s+1} \tag{2.11}$$

where $e_{p\,q}$ are $r \times r$ matrices such that $(e_{p\,q})_{ij} = \delta_{ip}\,\delta_{jq}$ and the phase factor V_0 is $2r \times 2r$ diagonal and unitary matrix.

The two Jost solutions are fundamental solutions and must be linearly dependent. This means that there exists a matrix $T(t,\lambda)$, called scattering matrix, which connects them and has an appropriate block structure.

$$T(t,\lambda) = \psi^{-1}(x,t,\lambda)\,\phi(x,t,\lambda) \tag{2.12}$$

The spectral parameter λ takes values in two-sheeted Riemannian surface $\mathcal{S}$:

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$$

associated with the square root $j(\lambda)$. Each sheet of this surface is determined by the sign of $j(\lambda)$.

$$\mathcal{S}_1 \colon \operatorname{Im} j(\lambda) > 0, \qquad \mathcal{S}_2 \colon \operatorname{Im} j(\lambda) < 0. \tag{2.13}$$

Half of the columns of the Jost solutions are analytic functions of λ on the first sheet and the other on the second sheet.

$$\psi(x,\lambda) = (|\psi^-(x,\lambda)\rangle, |\psi^+(x,\lambda)\rangle), \qquad \phi(x,\lambda) = (|\phi^+(x,\lambda)\rangle, |\phi^-(x,\lambda)\rangle), \tag{2.14}$$

where $|\psi^\pm\rangle$ and $|\psi^\pm\rangle$ denote a $r \times 2r$ matrix composed of the corresponding r columns of the Jost solutions. The superscript "$^+$" means analyticity on the first sheet and "$^-$" - analyticity on the second sheet. Next, we can construct FAS on each of the sheets by simply combining the blocks of the Jost solutions with the same analyticity properties.

$$\chi^+(x,\lambda) \equiv \left(|\phi^+\rangle, |\psi^+\rangle\right)(x,\lambda), \qquad \chi^-(x,\lambda) \equiv \left(|\psi^-\rangle, |\phi^-\rangle\right)(x,\lambda) \tag{2.15}$$

Let us write down the FAS $\chi^+(x,\lambda)$, analytic on the sheet $\mathcal{S}_1$ and $\chi^-(x,\lambda)$, analytic on the sheet $\mathcal{S}_2$ using appropriate decompositions of the scattering matrix (2.19), which consists of the same upper (lower) block-triangular functions $\mathbf{S}^\pm$ and $\mathbf{T}^\pm$ as they are in the v.b.c. case [8]:

$$\chi^\pm(x,\lambda) = \psi(x,\lambda)\mathbf{T}^\mp = \phi(x,\lambda)\mathbf{S}^\pm \tag{2.16}$$

These triangular factors are:

$$\mathbf{S}^+ = \begin{pmatrix} 1\!\!1 & \mathbf{d}^- \\ 0 & \mathbf{c}^+ \end{pmatrix}, \quad \mathbf{T}^- = \begin{pmatrix} \mathbf{a}^+ & 0 \\ \mathbf{b}^+ & 1\!\!1 \end{pmatrix}, \qquad \mathbf{S}^- = \begin{pmatrix} \mathbf{c}^- & 0 \\ -\mathbf{d}^+ & 1\!\!1 \end{pmatrix}, \qquad \mathbf{T}^+ = \begin{pmatrix} 1\!\!1 & -\mathbf{b}^- \\ 0 & \mathbf{a}^- \end{pmatrix} \tag{2.17}$$

and can be viewed also as generalized Gauss decompositions of the $T(\lambda)$.

$$T(\lambda) = \mathbf{T}^-(\lambda)\widehat{\mathbf{S}}^+(\lambda) = \mathbf{T}^+(\lambda)\widehat{\mathbf{S}}^-(\lambda). \tag{2.18}$$

Here and after the hat $\hat{}$ means taking the inverse matrix. We can use for the scattering matrix the same block-matrix structure as in v.b.c. case [11]:

$$\phi(x,\lambda) = \psi(x,\lambda)T(\lambda), \quad T(\lambda) = \begin{pmatrix} \mathbf{a}^+(\lambda) & -\mathbf{b}^-(\lambda) \\ \mathbf{b}^+(\lambda) & \mathbf{a}^-(\lambda) \end{pmatrix} \qquad \widehat{T}(\lambda) = \begin{pmatrix} \mathbf{c}^-(\lambda) & \mathbf{d}^-(\lambda) \\ -\mathbf{d}^+(\lambda) & \mathbf{c}^+(\lambda) \end{pmatrix} \tag{2.19}$$

The elements of the inverse matrix are defined as follows:

$$\begin{aligned} \mathbf{c}^-(\lambda) &= \widehat{\mathbf{a}}^+(\lambda)(1\!\!1 + \rho^-\rho^+)^{-1} = (1\!\!1 + \tau^+\tau^-)^{-1}\widehat{\mathbf{a}}^+(\lambda) \\ \mathbf{d}^-(\lambda) &= \widehat{\mathbf{a}}^+(\lambda)\rho^-(\lambda)\,(1\!\!1 + \rho^+\rho^-)^{-1} = (1\!\!1 + \tau^+\tau^-)^{-1}\tau^+(\lambda)\widehat{\mathbf{a}}^-(\lambda) \\ \mathbf{c}^+(\lambda) &= \widehat{\mathbf{a}}^-(\lambda)(1\!\!1 + \rho^+\rho^-)^{-1} = (1\!\!1 + \tau^+\tau^-)^{-1}\widehat{\mathbf{a}}^-(\lambda) \\ \mathbf{d}^+(\lambda) &= \widehat{\mathbf{a}}^-(\lambda)\rho^+(\lambda)\,(1\!\!1 + \rho^-\rho^+)^{-1} = (1\!\!1 + \tau^-\tau^+)^{-1}\tau^-(\lambda)\widehat{\mathbf{a}}^+(\lambda) \end{aligned}$$

Here $\rho^\pm(\lambda) = \mathbf{b}^\pm(\lambda)\widehat{\mathbf{a}}^\pm(\lambda) = \widehat{\mathbf{c}}^\pm(\lambda)\mathbf{d}^\pm(\lambda)$ and $\tau^\pm(\lambda) = \widehat{\mathbf{a}}^\pm(\lambda)\mathbf{b}^\mp(\lambda) = \mathbf{d}^\mp(\lambda)\widehat{\mathbf{c}}^\pm(\lambda)$ are the multicomponent generalizations of the reflection $\rho^\pm$, $\tau^\pm$ coefficients. (for the scalar case see [21, 22, 19]).

Given the potential $Q(x)$ one can obtain the Jost solutions uniquely. The Jost solutions in turn determine uniquely the scattering matrix $T(\lambda)$ and its inverse $\widehat{T}(\lambda)$. $Q(x)$ contains at most $|\triangle_+^1|$ independent complex-valued functions of x. Thus it is natural to expect that at most $|\triangle_1^+|$ of the coefficients of $T(\lambda)$ for $\lambda \in \mathbb{R}_m$, instead of $(2r)^2$, will be independent. Here $|\triangle_1^+|$ is the number of roots in $\triangle_1^+$,i.e. $|\triangle_1^+| = r(r+1)/2$ for $\mathbf{C}_r$ and $|\triangle_1^+| = r(r-1)/2$ for $\mathbf{D}_r$. The continuous spectrum $\mathbb{R}_m = (-\infty, -m) \cup (m, \infty)$ is determined by the condition $|\lambda| \geq m$.

The set of independent coefficients of $T(\lambda)$ are known as the set of minimal scattering data $\mathcal{T}$. They were introduced by Kaup for the Z-Sh system associated with $\mathfrak{g} \simeq sl(2)$ and v.b.c.. He proved that $a^\pm(\lambda)$ can be recovered from $\mathcal{T}$ using the analyticity properties, i.e. the so-called dispersion relation. The same problem for the generalized Z-Sh system with c.b.c. is more difficult. Here we just introduce $\mathcal{T}_i = \mathcal{T}_{i,c} \cup \mathcal{T}_{i,d}$ as the proper generalization of the minimal set of scattering data:

$$\begin{aligned} \mathcal{T}_{1,c} &\equiv \{\rho_\alpha^+(\lambda), \rho_\alpha^-(\lambda), \quad \lambda \in \mathbb{R}_m\}, \quad \mathcal{T}_{1,d} \equiv \{\rho_\alpha^\pm(\lambda_j^\pm), \lambda_j^\pm\}_{j=1}^N \\ \mathcal{T}_{1,c} &\equiv \{\tau_\alpha^+(\lambda), \tau_\alpha^-(\lambda), \quad \lambda \in \mathbb{R}_m\}, \quad \mathcal{T}_{2,d} \equiv \{\tau_\alpha^\pm(\lambda_j^\pm), \lambda_j^\pm\}_{j=1}^N \end{aligned}$$

where $\alpha \in \triangle_1^+$. The reconstruction of the diagonal blocks $\mathbf{a}^\pm(\lambda)$ from their analyticity properties requires a solution of $r \times r$ matrix-valued Riemman-Hilbert problem. Here $\lambda_j^\pm$ are discrete eigenvalues of L. The sets $\mathcal{T}_{i,c}$ characterizing the continuous spectrum need to be completed by the sets $\mathcal{T}_{i,d}$ characterizing the discrete spectrum of L which in turn requires the knowledge of the dressing factors. These problems will be addressed elsewhere.

3 Wronskian relations

Let the class of allowed potentials $\mathcal{M}$ be a slice of O_{σ_3} determined by additional constraints: i.) any generic element $F(x) = P_{\sigma_3}F(x)$ of $\mathcal{M}$ is matrix-valued function which vanishes fast enough

for $|x| \to \infty$ and ii.) the phase factor V which connect the asymptotic values of the potential $Q_+ = V^\dagger Q_- V$ is an integral of motion. The derivative of the potential $Q_x(x,t)$ belongs to the class of allowed potentials. The variation of the potential $\delta Q(x,t)$ is an allowed potential provided it satisfies the second additional condition. The mapping $\mathcal{F} : \mathcal{M} \to \mathcal{L}$ between the class of allowed potentials $\mathcal{M}$ and the scattering data $\mathcal{L}$ of L is analyzed by means of Wronskian relations [3, 4]. These relations allow us to formulate the main result of this work, i.e. that the ISM is a generalized Fourier transform in the case of **C.I** and **D.III**-type symmetric spaces. They also serve to introduce the skew-scalar product

$$\left[\!\left[A(x), B(x) \right]\!\right] = \frac{1}{2} \int dx \; \langle \, A(x) \, , \, [\, \sigma_3 \, , B(x) \,] \, \rangle \tag{3.1}$$

which is non-degenerate for $A(x), B(x) \in \mathcal{M}$ and provides it with symplectic structure. We start with the identity:

$$\langle \widehat{\chi} \; (Q(x,t) - \lambda \sigma_3) \; \chi(x,\lambda) \, , \, E_{\pm\alpha} \, \rangle \, |_{x=-\infty}^{\infty} = -i \int_{-\infty}^{\infty} dx \; \left\langle \, \frac{i}{2} \, [\, \sigma_3 \, , \sigma_3 \, Q_x] \, , \, P_{\sigma_3} \, \chi \, E_{\pm\alpha} \, \widehat{\chi}(x,\lambda) \right\rangle \tag{3.2}$$

where $\chi(x,\lambda)$ can be any fundamental solution of L. For convenience we choose them to be the FAS introduced above. The l.h.side of (3.2) can be calculated explicitly by using the asymptotics of FAS for $x \to \pm\infty$. It would be expressed by the matrix elements of the scattering matrix $T(\lambda)$, i.e. by the scattering data of L as follows:

$$\begin{aligned}
\left[\!\left[P_{\sigma_3} \, \chi^+(x,\lambda) \, E_\alpha \, \widehat{\chi}^+, \sigma_3 \, Q_x \right]\!\right] &= -j(\lambda) \left\langle \widehat{\mathbf{T}}^- \sigma_3 \mathbf{T}^- \, , \, E_\alpha \right\rangle = 2j(\lambda) \mathbf{b}_\alpha^+, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^+(x,\lambda) \, E_{-\alpha} \, \widehat{\chi}^+, \sigma_3 \, Q_x \right]\!\right] &= j(\lambda) \left\langle \widehat{\mathbf{S}}^+ \sigma_3 \mathbf{S}^+ \, , \, E_{-\alpha} \right\rangle = 2j(\lambda) \mathbf{d}_{-\alpha}^-, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^-(x,\lambda) \, E_\alpha \, \widehat{\chi}^-, \sigma_3 \, Q_x \right]\!\right] &= j(\lambda) \left\langle \widehat{\mathbf{S}}^- \sigma_3 \mathbf{S}^- \, , \, E_\alpha \right\rangle = 2j(\lambda) \mathbf{d}_\alpha^+, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^-(x,\lambda) \, E_{-\alpha} \, \widehat{\chi}^-, \sigma_3 \, Q_x \right]\!\right] &= -j(\lambda) \left\langle \widehat{\mathbf{T}}^+ \sigma_3 \mathbf{T}^+ \, , \, E_{-\alpha} \right\rangle = 2j(\lambda) \mathbf{b}_{-\alpha}^-, \qquad && \alpha \in \triangle_1^+
\end{aligned} \tag{3.3}$$

The second set of Wronskian relations which we consider relate the variation of the potential δQ to the corresponding variations of the scattering data $\delta\rho$ and $\delta\tau$. For this purpose we use the identity:

$$\langle \widehat{\chi} \, \delta\chi(x,\lambda) \, , \, E_{\pm\alpha} \, \rangle \, |_{x=-\infty}^{\infty} = \int_{-\infty}^{\infty} dx \; \left\langle \, \frac{i}{2} \, [\, \sigma_3 \, , \sigma_3 \, \delta Q] \, , \, P_{\sigma_3} \, \chi(x,\lambda) \, E_{\pm\alpha} \, \widehat{\chi} \right\rangle \tag{3.4}$$

If we assume that the variation of the phase factor δV vanishes we arrive at:

$$\begin{aligned}
\left[\!\left[P_{\sigma_3} \, \chi^+(x,\lambda) \, E_\alpha \, \widehat{\chi}^+, \sigma_3 \, \delta Q \, \right]\!\right] &= -i \left\langle \widehat{\mathbf{T}}^- \delta \mathbf{T}^- \, , \, E_\alpha \right\rangle = i(\delta\rho^+ \mathbf{a}^+)_\alpha, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^+(x,\lambda) \, E_{-\alpha} \, \widehat{\chi}^+, \sigma_3 \, \delta Q \, \right]\!\right] &= i \left\langle \widehat{\mathbf{S}}^+ \delta \mathbf{S}^+ \, , \, E_{-\alpha} \right\rangle = i(\delta\tau^+ \mathbf{c}^+)_{-\alpha}, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^-(x,\lambda) \, E_\alpha \, \widehat{\chi}^-, \sigma_3 \, \delta Q \, \right]\!\right] &= i \left\langle \widehat{\mathbf{S}}^- \delta \mathbf{S}^- \, , \, E_\alpha \right\rangle = i(\delta\tau^- \mathbf{c}^-)_\alpha, \qquad && \alpha \in \triangle_1^+ \\
\left[\!\left[P_{\sigma_3} \, \chi^-(x,\lambda) \, E_{-\alpha} \, \widehat{\chi}^-, \sigma_3 \, \delta Q \, \right]\!\right] &= -i \left\langle \widehat{\mathbf{T}}^+ \delta \mathbf{T}^+ \, , \, E_{-\alpha} \right\rangle = i(\delta\rho^- \mathbf{a}^-)_{-\alpha}, \qquad && \alpha \in \triangle_1^+
\end{aligned} \tag{3.5}$$

These relations are basic for the analysis of the related NLEE and their Hamiltonian structures. The above identities also allow us to introduce the proper generalizations of the usual Fourier exponential functions. Let us introduce the set of "squared solutions":

$$\Phi_\alpha^\pm(x,\lambda) = P_{\sigma_3} \, \chi^\pm(x,\lambda) E_{\pm\alpha} \widehat{\chi}^\pm(x,\lambda) \, , \quad \text{for } \alpha \in \triangle_1^+ \tag{3.6}$$

$$\Psi_\alpha^\pm(x,\lambda) = P_{\sigma_3}\,\chi^\pm(x,\lambda)E_{\mp\alpha}\widehat{\chi}^\pm(x,\lambda)\,, \quad \text{for } \alpha\in\triangle_1^+ \tag{3.7}$$

$$\Theta_\alpha^\pm(x,\lambda) = P_{\sigma_3}\,\chi^\pm(x,\lambda)E_{\pm\alpha}\widehat{\chi}^\pm(x,\lambda)\,, \quad \text{for } \alpha\in\triangle_0^+ \tag{3.8}$$

$$\Xi_\alpha^\pm(x,\lambda) = P_{\sigma_3}\,\chi^\pm(x,\lambda)E_{\mp\alpha}\widehat{\chi}^\pm(x,\lambda)\,, \quad \text{for } \alpha\in\triangle_0^+ \tag{3.9}$$

$$\Upsilon_k^\pm(x,\lambda) = P_{\sigma_3}\,\chi^\pm(x,\lambda)H_k\widehat{\chi}^\pm(x,\lambda)\,, \quad \text{for } k=1,...,r \tag{3.10}$$

These are the "squared solutions" of the Lax operator L connected with simple Lie algebra $\mathfrak{g}$. They are constructed by means of FAS $\chi^\pm(x,\lambda)$ and the Cartan-Weyl basis of the algebra and are analytic functions of λ on the corresponding sheets of the spectral surface. The equations that $\Phi_\alpha^\pm$ an $\Psi_\alpha^\pm$ satisfy are a direct consequence of the fact that FAS and their inverse satisfy the Z-Sh system system:

$$i\frac{d\,\Phi_\alpha^\pm}{dx} + \left[Q(x)-\lambda\,\sigma_3, \Phi_\alpha^\pm(x,\lambda)\right] = 0, \qquad i\frac{d\,\Psi_\alpha^\pm}{dx} + \left[Q(x)-\lambda\,\sigma_3, \Psi_\alpha^\pm(x,\lambda)\right] = 0 \tag{3.11}$$

The "squared solutions" also serve as building blocks of the Green function for L [6, 7, 8]:

$$\mathbf{G}^\pm(x,y,\lambda) = G_1^\pm(x,y,\lambda)\theta(y-x) - G_2^\pm(x,y,\lambda)\theta(x-y), \tag{3.12}$$

where

$$G_1^\pm(x,y,\lambda) = \sum_{\alpha\in\triangle_1^+}\Phi_\alpha^\pm(x,\lambda)\otimes\Psi_\alpha^\pm(y,\lambda) \tag{3.13}$$

$$G_2^\pm(x,y,\lambda) = \sum_{\alpha\in\triangle_1^+}\Psi_\alpha^\pm(x,\lambda)\otimes\Phi_\alpha^\pm(y,\lambda) + \sum_{\alpha\in\triangle_0^+\cup\triangle_1^+}\Xi_\alpha^\pm(x,\lambda)\otimes\Theta_\alpha^\pm(y,\lambda) + \sum_{k=1}^{r}\Upsilon_k^\pm(x,\lambda)\otimes\Upsilon_k^\pm(y,\lambda) \tag{3.14}$$

and $\theta(x)$ is the usual step function.

4 Completeness relation and evolution

The main result in this section is that the sets $\{\Phi_\alpha^\pm\}$ and $\{\Psi_\alpha^\pm\}$ form complete sets of functions in $\mathcal{M}$. The idea of the proof is simple. Apply the contour integration method along a proper contour (see figure.1) to a conveniently chosen Green function (3.12). From the Cauchy theorem we have

$$\frac{1}{2\pi i}\oint_C \mathbf{G}^+(x,y,\lambda)d\lambda = \sum_{k=1}^{N}\operatorname*{Res}_{\lambda=\lambda_k^\pm}\mathbf{G}^+(x,y,\lambda) \tag{4.1}$$

Integrating along the contours we treat separately the contribution from the infinite semi-arcs and the ones from the continuous spectrum $R_m = C_1\cup C_2$ which is composed of the cuts $C_1 = (-\infty,-m)$ and $C_2 = (m,\infty)$. Special care must be taken for the end points $\lambda = \pm m$ of the spectrum. Assuming that the end points of the spectrum give no contribution

$$\lim_{\varepsilon\to 0}\left(\int_{\varepsilon_+} + \int_{\varepsilon_-}\right)\mathbf{G}^+ d\lambda = 0 \tag{4.2}$$

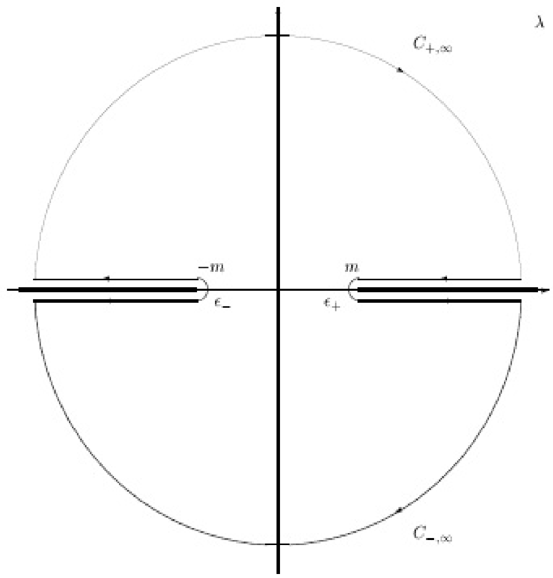

Figure 1: The contour along which we integrate lies completely on the first sheet of the 2-sheeted spectral surface associated with the square root $j(\lambda) = \sqrt{\lambda^2 - m^2}$

where with index $\varepsilon_\pm$ we have denoted the integrals along the infinitesimal semi-arcs around the end points of the spectrum, we obtain the following completeness relation

$$\begin{aligned}\delta(x-y)\Pi_{\sigma_3} &= \frac{1}{\pi}\sum_{\alpha\in\triangle_1^+}\int_{R_m} d\lambda\, \left\{\Phi_\alpha^+(x,\lambda)\otimes\Psi_\alpha^+(y,\lambda) - \Phi_\alpha^-(x,\lambda)\otimes\Psi_\alpha^-(y,\lambda)\right\} \\ &- 2i\sum_{\alpha\in\triangle_1^+}\sum_{k=1}^{N}\left\{\frac{d}{d\lambda}\Phi_\alpha^+(x,\lambda)\Big|_{\lambda=\lambda_k^+}\otimes\Psi_\alpha^+(y,\lambda) + \Phi_\alpha^+(x,\lambda)\otimes\frac{d}{d\lambda}\Psi_\alpha^+(y,\lambda)\Big|_{\lambda=\lambda_k^+}\right\}\end{aligned}$$

Here $\Pi_{\sigma_3} = \sum_{\alpha\in\triangle_1^+}[E_\alpha\otimes E_{-\alpha} - E_{-\alpha}\otimes E_\alpha]$. The assumption that we have made is that λ_j^+ are simple poles of the "squared solutions" Φ_α^+ and Ψ_α^+.

Using the completeness relation one can expand any generic element of the phase space $\mathcal{M}$ over each of the complete sets of "squared solutions" $\Psi_\alpha^\pm$ and $\Phi_\alpha^\pm$. This relation is utilized with the help of the following the trick

$$-\frac{1}{2}\mathrm{tr}_1\left\{([\sigma_3, F(x)]\otimes 1\!\!1)\Pi_{\sigma_3}\right\} = \frac{1}{2}\mathrm{tr}_2\left\{\Pi_{\sigma_3}(1\!\!1\otimes[\sigma_3, F(x)])\right\} = F(x) \tag{4.3}$$

where tr_1 (and tr_2) mean taking the trace of the elements in the first (or in the second) position of the tensor product.

The completeness relation (4.3) allows to establish one-to-one correspondence between the elements of $\mathcal{M}$, such as Q_x and Q_t, and its expansion coefficients. It is also directly related to the spectral decompositions of the generating (recursion) operators $\Lambda_\pm$. These operators are the ones whose eigenfunctions are the "squared solutions". Their derivation starts by introducing the splitting of the object $e_\alpha^\pm = \chi^\pm(x,\lambda)E_{\pm\alpha}\widehat{\chi}^\pm(x,\lambda)$ into block diagonal and block off-diagonal parts:

$$e_\alpha^\pm(x,\lambda) = e_\alpha^{d,\pm}(x,\lambda) + \Phi_\alpha^\pm(x,\lambda), \qquad e_\alpha^{d,\pm}(x,\lambda) = (1\!\!1 - P_{\sigma_3})\, e_\alpha^\pm(x,\lambda) \tag{4.4}$$

end use the equation it satisfies

$$i\frac{d\,e_{\alpha}^{\pm}}{dx} + \left[Q(x) - \lambda\,\sigma_3, e_{\alpha}^{\pm}(x,\lambda)\right] = 0 \tag{4.5}$$

Thus equation (4.5) splits into

$$i\frac{d\,e_{\alpha}^{d,\,\pm}}{dx} + \left[Q(x), \Phi_{\alpha}^{\pm}(x,\lambda)\right] = 0 \tag{4.6}$$

$$i\frac{d\,\Phi_{\alpha}^{\pm}}{dx} + \left[Q(x), e_{\alpha}^{d,\,\pm}(x,\lambda)\right] = \lambda\left[\sigma_3, \Phi_{\alpha}^{\pm}(x,\lambda)\right] \tag{4.7}$$

Equation (4.6) can be integrated formally with the result

$$e_{\alpha}^{d,\,\pm}(x,\lambda) = C_{\alpha;\,\epsilon}^{d,\,\pm}(\lambda) + i\int_{\epsilon\infty}^{x} dy\,\left[Q(y), \Phi_{\alpha}^{\pm}(y,\lambda)\right], \tag{4.8}$$

$$C_{\alpha;\,\epsilon}^{d,\,\pm}(\lambda) = \lim_{x\to\epsilon\infty} e_{\alpha}^{d,\,\pm}(x,\lambda), \qquad \epsilon = \pm 1 \tag{4.9}$$

Next insert (4.8) into (4.7) and act on both sides by $\mathrm{ad}_{\sigma_3}^{-1}$. This gives us:

$$(\Lambda_{\pm} - \lambda)\,\Phi_{\alpha}^{\pm}(x,\lambda) = i\left[C_{\alpha;\,\epsilon}^{d,\,\pm}(\lambda), \mathrm{ad}_{\sigma_3}^{-1}Q(x)\right], \tag{4.10}$$

where the generating operators $\Lambda_{\pm}$ are given by:

$$\Lambda_{\pm}\Xi(x) = \mathrm{ad}_{\sigma_3}^{-1}\left\{i\frac{d\Xi}{dx} + i\left[Q(x)\,,\,\int_{\pm\infty}^{x} dy\,\left[Q(y)\,,\,\Xi(y)\right]\right]\right\} \tag{4.11}$$

Thus $\Psi_{\alpha}^{\pm}$ (resp. $\Phi_{\alpha}^{\pm}$) will be eigenfunctions of Λ_{+} (resp. Λ_{-}) if and only if $C_{\alpha;\,\epsilon}^{d,\,\pm}(\lambda) = 0$. Evaluating the limit of (4.9) for all α in the specific case (4.2) we find:

$$(\Lambda_{+} - \lambda)\,\Psi_{\alpha}^{\pm}(x,\lambda) = 0 \qquad \left(\Lambda_{+} - \lambda_j^{\pm}\right)\Psi_{\alpha}^{\pm}(x,\lambda_j^{\pm}) = 0, \qquad \alpha \in \triangle_1^{+} \tag{4.12}$$

$$(\Lambda_{-} - \lambda)\,\Phi_{\alpha}^{\pm}(x,\lambda) = 0 \qquad \left(\Lambda_{-} - \lambda_j^{\pm}\right)\Phi_{\alpha}^{\pm}(x,\lambda_j^{\pm}) = 0, \qquad \alpha \in \triangle_1^{+} \tag{4.13}$$

This result can be generalized for arbitrary $f(\Lambda_{\pm})$:

$$(f(\Lambda_{+}) - f(\lambda))\,\Psi_{\alpha}^{\pm}(x,\lambda) = 0 \qquad \left(f(\Lambda_{+}) - f(\lambda_j^{\pm})\right)\Psi_{\alpha}^{\pm}(x,\lambda_j^{\pm}) = 0, \qquad \alpha \in \triangle_1^{+} \tag{4.14}$$

$$(f(\Lambda_{-}) - f(\lambda))\,\Phi_{\alpha}^{\pm}(x,\lambda) = 0 \qquad \left(f(\Lambda_{-}) - f(\lambda_j^{\pm})\right)\Phi_{\alpha}^{\pm}(x,\lambda_j^{\pm}) = 0, \qquad \alpha \in \triangle_1^{+} \tag{4.15}$$

The class of higher MNLS on symmetric spaces of **C.I** and **D.III**-type and with c.b.c. can be put down in terms of the derivative of the potential Q_t with respect to the evolution parameter and the dispersion law $f(\lambda) = -2\lambda$ [18, 8] as follows:

$$i\mathrm{ad}_{\sigma_3}^{-1}\frac{\partial}{\partial t}Q + f(\Lambda)\mathrm{ad}_{\sigma_3}^{-1}Q_x = 0 \tag{4.16}$$

Substituting the objects in this formula with their expansions over the "squared solutions" we obtain equations for the evolution of the scattering data. The expansion coefficients of $\mathrm{ad}_{\sigma_3}^{-1}Q_t$ and $\mathrm{ad}_{\sigma_3}^{-1}Q_x$ on the continuous spectrum turn out to be exactly the minimal set of scattering data. The evolution for the reflection and transition coefficients is provided by

$$i\frac{\partial\rho^{\pm}}{\partial t} \pm f(\lambda)\,j(\lambda)\rho^{\pm}(t,\lambda) = 0, \qquad i\frac{\partial\tau^{\pm}}{\partial t} \mp f(\lambda)\,j(\lambda)\tau^{\pm}(t,\lambda) = 0 \qquad \lambda \in \mathbb{R}_m. \tag{4.17}$$

The observation that the scattering data evolves trivially is visible from the equation depicting the evolution of the scattering matrix $T(\lambda)$. This equation is a result of the compatibility condition (1.8) and the fact that the two Jost solutions ψ and ϕ are solutions of the second operator of the Lax pair in the Z-Sh system (1.9). Acting with $i\frac{d}{dt}$ on $T(\lambda)$ (2.12), we get:

$$i\frac{d}{dt}T(t,\lambda) - 2\lambda j(\lambda)\,[\,\sigma_3, T(\lambda)] = 0, \tag{4.18}$$

where $f(\lambda) = -2\lambda$ is the dispersion law for the MNLS with c.b.c.. This equation for MNLS can also be derived from the explicit form of the Lax representation (1.10) by evaluating the limit $\lim_{x\to\pm\infty} M\psi = 0$. For the $r \times r$ blocks making up the scattering matrix we have:

$$\frac{\partial \mathbf{a}^{\pm}(t,\lambda)}{\partial t} = 0, \qquad i\frac{\partial \mathbf{b}^{\pm}}{\partial t} \mp 2\lambda\, j(\lambda)\mathbf{b}^{\pm}(t,\lambda) = 0 \tag{4.19}$$

These equations have obvious solutions. From the first equation is clear that the diagonal blocks are conserved and their invariants - upper (lower) principal minors as well as their determinants are generating functionals of the special series of local infinitely many integrals of motion I_k:

$$\ln\det\mathbf{a}^{\pm}(\lambda) = \sum_{k=1}^{\infty} \lambda^{-k} I_k \tag{4.20}$$

This is the major idea of the ISM - a one-to-one change of variables- from the multicomponent $\mathbf{q}(x,t)$, in terms of which the MNLS(1.15) is written, towards the scattering data which satisfy linear evolution equations.

5 Conclusion

The result of this work is that the interpretation of the ISM as a generalized Fourier transformation holds true in the case of Lax operators with constant boundary conditions on symmetric spaces connected with the Lie algebras $\mathbf{C}_r \simeq sp(2r)$ and $\mathbf{D}_r \simeq so(2r)$. The completeness relation of the "squared solutions" of the generalized Z-Sh system in the case when the Lax operator L becomes self-adjoint is derived. The "squared solutions" turn out to be generalizations of the usual Fourier exponential function and eigenfunctions of the recursion operators $\Lambda_{\pm}$. This result allows one to prove that the corresponding NLEE results in linear evolution for the scattering data. The recursion operators $\Lambda_{\pm}$ open the path towards the construction of *action-angle* variables for the NLEE solvable with this generalization of the Z-Sh system and from there the Hamiltonian formulation of these equations and their hierarchies connected with $\Lambda_{\pm}$.

The physical applications of the NLS eq. both with vanishing and non-vanishing boundary conditions is well known; the same holds true for the Manakov system as well as for the $sp(4)$ MNLS with v.b.c., see [13]. It will be interesting to find physical applications also for the MNLS with c.b.c.

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Appendix

The above definition of $\mathfrak{g}$ (2.4) satisfies the requirement that the Cartan subalgebra $\mathfrak{h}$ will be made up of diagonal matrices. The Cartan generators H_k, dual to e_k, are given by:

$$H_k = E_{kk} - E_{\bar{k}\,\bar{k}} \tag{5.1}$$

The element $\sigma_3 = \sum_{k=1}^{r} H_k$, belongs to $\mathfrak{h}$ and is dual to $\vec{a}$.

The root vectors in the typical representation are given by

$$E_{e_i - e_j} = E_{ij} - (-1)^{i+j} E_{\bar{j}\,\bar{i}} \qquad E_{e_i + e_j} = E_{i\,\bar{j}} - \epsilon_0 (-1)^{i+j} E_{j\,\bar{i}} \tag{5.2}$$

where $1 \leq i < j \leq r$ and $\epsilon_0 = \pm 1$. Since $\epsilon_0 = 1$ for $\mathfrak{g} \simeq so(2r)$ equation (5.2) gives vanishing result for $i = j$ which is compatible with the fact that $2e_i$ are not roots of $so(2r)$; for $\mathfrak{g} \simeq sp(2r)$ $\epsilon_0 = -1$ and equation (5.2) by putting $i = j$ provides also an expression for E_{2e_i}. However this expression is not normed with respect to the Killing form $\langle E_\alpha, E_{-\alpha} \rangle = 2$. The Weyl generators associated with the root $2e_i$ that we will use are given by [12]:

$$E_{2e_i} = \sqrt{2}\, E_{i\,\bar{i}} \tag{5.3}$$

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